6.1 Past and Future

The notion that events proceed in a ell defined sequence is unquestionable in classical mechanics. Events occur one after the other, and our kno ledge concerning the events at one time allo s us to predict hat ill occur at another time. One can unambiguously determine hether events lie in the past or future of other events. Given t o events, A and B, one can compute hich event occurred first. It may be, that event A causes event B, in hich case, event A must have preceded event B.

In this chapter, e are interested in hether the ell defined classical concepts of temporal ordering have a quantum analogue. In other ords, given t o quantum mechanical systems, can e measure hich system attains a particular state first. Can e decide hether an event occurs in the past or future of another event. The problem of measuring the time ordering of t o events is in some sense more primitive and fundamental a concept than that of measuring the time of an event.

We sa previously that one cannot measure the time-of-arrival to an accuracy better than $1/\bar{E}_k$ here $\bar{E}_k$ is the kinetic energy of the particle. This leads one to suspect that the ordering of events may not be an unambiguous concept in quantum mechanics. Ho ever, for a single quantum event A, although one cannot determine the time an event occurred to arbitrary accuracy, it can be argued that one can often measure hether A occurred before or after a fixed time t_B to any desired precession.

Consider a quantum system initially prepared in a state ψ_A and an event A hich corresponds to some projection operator Π_A acting on this state. For example, e could initially prepare an atom in an excited state, and Π_A could represent a projection onto all states here the atom is in its ground state i.e. the atom has decayed. ψ_A could also represent a particle localized in the region $x < 0$ and Π_A could be a projection onto the positive x-axis. In this case, the event A corresponds to the particle arriving to $x = 0$.

If the state evolves irreversible to a state for hich $\Pi_A \Psi(t) = 1$, then e can easily measure hether the event A has occurred at any time t. We could therefore measure hether a free particle arrives to a given location before or after a classical time t_B. Of course, for many systems, the system ill not irreversible evolve to the required state. For example, a particle influenced by a potential may cross over the origin many times. Ho ever, for an event such as atomic decay, the probability of the atom being re-excited is relatively small, and one can argue that the event is more or less irreversible.

For the case of a free particle hich is traveling to ards the origin from $x < 0$ one can argue that if at a later time I measure the projection operator onto the positive axis and find it there, then the particle must have arrived to the origin at some earlier time. This is in some sense a definition, because e kno of no ay to measure the particle being at the origin ithout altering its evolution or being extremely lucky and happening to measure the particle's location hen it is at the origin.

While measuring hether an event happened before or after a fixed time t_B may be possible, e ill find that for t o quantum systems, one cannot in general measure hether the time t_A of event A, occurred before or after the time t_B of event B.

In Section 6.2, confining ourselves to a particular example of order of events, e ill consider the question of order of arrival in quantum mechanics. Given t o particles, can e determine hich particle arrived first to the location x_a. Using a model detector, e find that there is al ays an inherent inaccuracy in this type of measurement, given by $1/\bar{E}$ here $\bar{E}$ is the typical total energy of the t o particles. This seems to suggest that the notion of past and future is not a ell defined observable in quantum mechanics.

Note that the measurements e are considering here are continuous measurements, as opposed to the impulsive measurement of an operator. One could, for example, determine the order of arrival by measuring the operator

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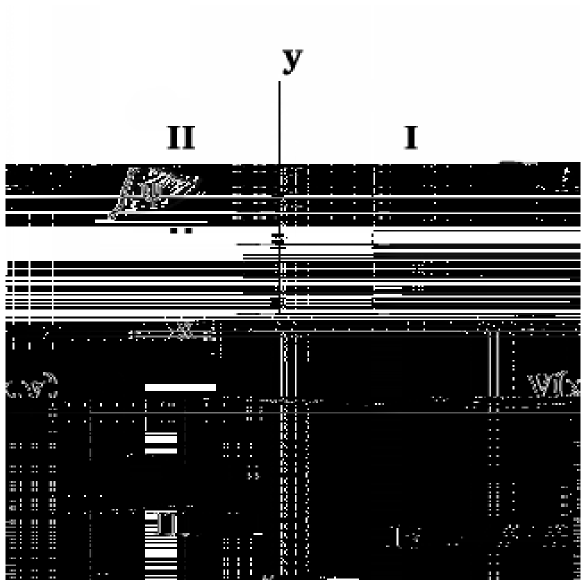


Figure 6.3: A potential hich can be used to measure hich of t o particles came first (given by $V(x, y) = \alpha\delta(\mathbf{x})\theta(-\mathbf{y})$). The ave function for t o incoming particles in one dimension looks like a single ave packet in t o dimensions travelling to ards the origin.

sees the potential, and both particles ill continue traveling past the origin. One can therefore ait a sufficiently long period of time, and measure here the t o particles are. If both the x and y particles are found past the origin, then e kno that the x-particle arrived first. If the y-particle is found past the origin hile the x-particle has been reflected back into the positive x-axis then e kno that the y-particle arrived first.

Classically, this method ould appear to unambiguously measure hich of the t o particle arrived first. Ho ever, in quantum mechanics, this method fails. From (6.190) e can see that the problem of measuring hich particle arrives first is equivalent to deciding here a single particle traveling in a plane arrives. T o particles localized to the right of the origin is equivalent to a single particle localized in the first quadrant (see Figure 6.3). The question of hich particle arrives first, becomes equivalent to the question of hether the particle crosses the positive x-axis or the positive y-axis.

The Hamiltonian (6.191) is therefore equivalent to the problem of scattering off a

to the origin (the sharp edge of the potential). The amplitude for being scattered off the region around the edge in the direction θ is given by $|f(r,\theta)|^2$.

It might be argued that since these particles scattered, they must have scattered off the potential, and therefore they represent experiments in hich the y-particle arrived first. Ho ever, this ould clearly over count the cases here the y-particle arrived first. We could have just as easily have placed our potential on the negative x-axis, in hich case, e ould over count the cases here the x-particle arrived first.

In the "interference region" e cannot have confidence that our measurement orked at all. We should therefore define a "failure cross section" given by

$$\begin{aligned} \sigma_f &= \int_0^{2\pi} |f(\theta)|^2 \\ &= \frac{1}{k\cos(\frac{\theta_o}{2})} \end{aligned} \tag{6.195}$$

From (6.195) e can see that cross section for scattering off the edge is the size of the particle's avelength multiplied by some angular dependence. Therefore, if the particle arrives ithin a distance of the origin given by

$$\delta x > 2/k \tag{6.196}$$

the measurement fails. We have dropped the angular dependence from (6.195) – the angular dependence is not of physical importance for measuring hich particle came

In other ords, our measurement procedure relies on making an inference bet een time measurements and spatial coordinates. The last t o equations then give us

$$\delta t > \frac{1}{E} \quad . \tag{6.198}$$

One ill not be able to determine hich particle arrived first, if they arrive ithin a time $1/E$ of each other, here E is the total kinetic energy of both particles. Note that Equation (6.198) is valid for a plane ave ith definite momentum k. For ave functions for hich $dk << k$, one can replace E by the expectation value $\langle E \rangle$. Ho ever, for ave functions hich have a large spread in momentum, or hich have a number of distinct peaks in k, then to ensure that the measurement almost al ays orks, one must measure the order of arrival ith an accuracy given by

$$\delta t > \frac{1}{\bar{E}} \tag{6.199}$$

here $\bar{E}$ is the minimum typical total energy [1].

Although it seemed plausible that one could measure hich particle arrived first, e found that if the particles are coincident to ithin $1/\bar{E}$, then the measurement fails.

6.3 Coincidence

In the previous model for measuring hich particle arrived first, e found that if the t o particles arrived to ithin $1/\bar{E}$ of each other, the measurement did not succeed. The idth $1/\bar{E}$ as an inherent inaccuracy hich could not be overcome. Ho ever, in our simple m

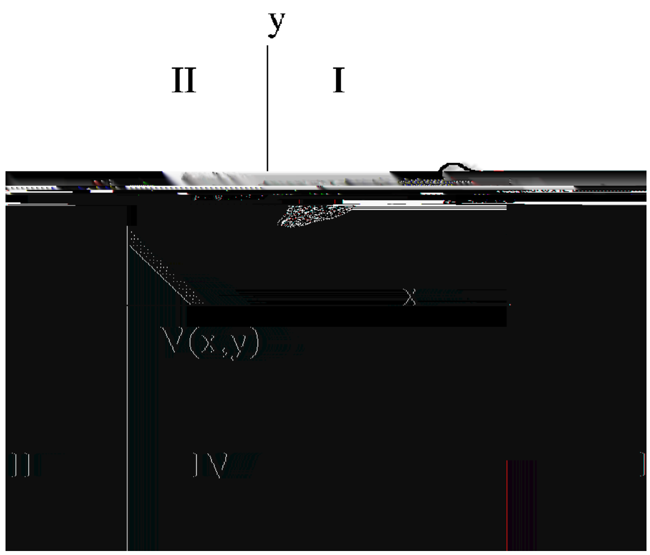


Figure 6.4: Potential for measuring hether t o particles are coincident.

traveling to ards the origin, e ask hether they arrive ithin a time δt_c of each other. If the particles do not arrive coincidently, then e do not concern ourselves ith hich arrived first. The parameter δt_c arro mechanically, e once again find an interference region around the strip hich scatters particles into the classically forbidden regions of quadrant t o and four. The shado is not sharp, and e are not al ays certain hether the particles ere coincident.

A solution to plane aves scattering off a narro strip is ell kno n and can be found in many quantum mechanical texts (see for example [46] here the scattered ave is ritten as a sum of products of Hermite polynomials and Mathieu functions). Ho ever, for our purposes, e ill find it convenient to consider a simpler model for measuring coincidence, namely, an infinite circular potential of radius a, centered at the origin.

$$H_i = \alpha V(r/a) \tag{6.200}$$

here $V(x)$ is the unit disk, and e take the limit $\alpha \to \infty$.

It is ell kno n that if $a < 1/k$, then there ill not be a ell-defined shado behind the disk. To see this, consider a plane ave coming in from negative x-infinity. It can be expanded in terms of the Bessel function $J_m(kr)$ and then ritten asymptotically ($r \gg 1$) as a sum of incoming and outgoing circular aves.

$$\begin{aligned} e^{ikx} &= \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos m\theta \\ &\simeq \sqrt{\frac{1}{2\pi ikr}} \left[e^{ikr} \sum_{m=0}^{\infty} \epsilon_m \cos m\theta + ie^{-ikr} \sum_{m=0}^{\infty} \epsilon_m \cos m(\theta - \pi) \right] \quad . \end{aligned} \tag{6.201}$$

here ϵ_m is the Neumann factor hich is equal to 1 for $m = 0$ and equal to 2 other ise.

Since it can be sho n that

$$\sum_{m=0}^{M} \epsilon_m \cos m\theta = \frac{\sin{(M + \frac{1}{2})\theta}}{\sin \frac{1}{2}\theta} \tag{6.202}$$

The t o infinite sums approach $2\pi\delta(\theta)$ and $2\pi\delta(\theta - \pi)$ respectively, and so the incoming ave comes in from the left, and the outgoing ave goes out to the right. The presence of the potential modifies the ave function and in addition to the plane ave, produces a scattered ave

$$\psi = e^{ikx} + \frac{e^{ikr}}{\sqrt{r}} f(r\theta) \tag{6.203}$$

here

$$\frac{e^{ikr}}{\sqrt{r}} f(r,\theta) = -i \sum_{m=0}^{\infty} \epsilon_m e^{\frac{1}{2}m\pi i - i\delta_m} \sin\delta_m H_m(kr) \cos m\theta \quad , \tag{6.204}$$

$H_m(kr)$ are Hermite polynomials and

$$\tan\delta_m = \frac{-J_m(ka)}{N_m(ka)} \tag{6.205}$$

($N_m(ka)$ are Bessel functions of the second kind). For large values of r, the ave function can be ritten in a manner similar to (6.201), except that the outgoing ave is modified by the phase shifts δ_m.

$$\psi \simeq \frac{1}{\sqrt{2\pi ik}} i \sum_{m=0}^{\infty} \epsilon_m \cos m(\theta - \pi) \frac{e^{-ikr}}{\sqrt{r}} + \frac{e^{ikr}}{\sqrt{r}} f(r,\theta) \quad . \tag{6.206}$$

here

$$f(r,\theta) \simeq \frac{1}{\sqrt{2\pi ik}} \sum_{m=0}^{\infty} \epsilon_m e^{-2i\delta_m(ka)} \cos m\theta \tag{6.207}$$

In the limit that $ka \gg m$ the phase shifts can be ritten as

$$\delta_m \simeq ka - \frac{\pi}{2}(m + \frac{1}{2}) \quad . \tag{6.208}$$

In the limit of extrememely large a (but $a < r$), the outgoing aves then behave as

$$f(r,\theta) \simeq \lim_{M\to\infty} -i \frac{1}{\sqrt{2\pi ik}} e^{-2ika} \frac{\sin{(M + \frac{1}{2})(\theta - \pi)}}{\sin\frac{1}{2}(\theta - \pi)} \tag{6.209}$$

here once again e see that the angular distribution goes as the delta function $\delta(\theta - \pi)$. The disk scatters the plane ave directly back, and a sharp shado is produced. We see therefore, that in the limit of $ka \gg 1$, our measurement of coincidence orks.

The differential cross section can in general be ritten as

$$\begin{aligned} \sigma &= |f(\theta)|^2 \\ &= |\sum_{m=0}^{\infty} \epsilon_m e^{-2i\delta_m(ka)} \cos m\theta|^2 \end{aligned} \tag{6.210}$$

For $ka \gg 1$ (but still finite), (6.210) can be computed using our expression for the phase shifts from (6.208), and is given by

$$\sigma(\theta) \simeq \frac{a}{2} \sin \frac{\theta}{2} + \frac{1}{2\pi k} \cot^2 \frac{\theta}{2} \sin^2 ka\theta \tag{6.211}$$

The first term represents the part of the plane ave hich is scattered back, hile the second term is a for ard scattered ave hich actually interferes ith the plane- ave. The reason it appears in our expression for the scattering cross section is because e have ritten our ave function as the sum of a plane- ave and a scattered ave, and so part of the scattered ave must interfere ith the plane- ave to produce the shado behind the disk.

For $ka \ll m$, the phase shifts look like

$$\delta_m(ka) \simeq \frac{\pi m}{(m!)^2} \left(\frac{ka}{2}\right)^{2m} \qquad m \neq 0 \tag{6.212}$$

and

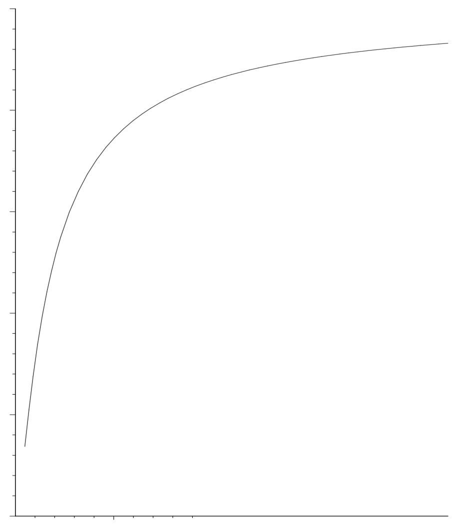
$$\tan \delta_0(ka) \simeq \frac{-\pi}{2 \ln ka} \tag{6.213}$$

As a result, for $ka \ll 1$, δ_0 is much greater than all the other δ_m and the outgoing solution is almost a pure isotropic s- ave.

For $ka \ll 1$ the only contribution to (6.210) comes from δ_0 and the differential cross section becomes

$$\sigma(\theta) \simeq \frac{\pi}{2k \ln^2 ka} \tag{6.214}$$

and is isotropic. In other ords, no shado is formed at all, and particles are scattered into classically forbidden regions. We see therefore, that as long as the s- ave is dominant, our measurement fails. The s- ave ill cease being dominant hen δ_0 is of the same order as δ_1. As can be seen from Equation 6.208, δ_1/δ_0 approaches a limiting value of 1 hen a sharp shado is produced. It is only hen $\delta_1/\delta_0 \simeq 1$ that the cross-section no longer



Preparing a state ψ_c corresponds to preparing a single particle in t o dimensions hich al ays arrives inside a region $\delta r = p\delta t_c/m$ of the origin. In other ords, suppose e ere to set up a detector of size δr at the origin. If a state ψ_c exists, then it ould al ays trigger the detector at some later time.

Our definition of coincidence requires that the state ψ_c not be a state here one particle arrives at a time $t > \delta t_c$ before the other particle. In other ords, if instead, e ere to perform a measurement on ψ_c to determine hether particle x arrived at least δt_c before particle y, then e must get a negative result for this measurement.

This latter measurement ould correspond to the t o-dimensional experiment of placing a series of detectors on the positive y-axis, and measuring hether any of them are triggered by ψ_c. If ψ_c is truly a coincident state, then none of the detectors hich are placed at a distance greater than $y = \delta r$ can be triggered. One could even consider a single detector, placed for example, at $(0, \delta r)$, and one ould require that ψ_c not trigger this detector.

determine hether the detector is at the origin, or at $(0, \delta r)$ if $\delta r < 2\pi/k$. As a result, ψ_c can only be coincident to a region around the origin of radius less than δr or, coincident ithin a time $\delta t_c \sim 1/E$.

It is also interesting to consider the situation here e have an event B hich must be preceded by an event A. For example, B could be caused by A, or the dynamics could be such that B can only occur hen the system is in the state A. One can then attempt to force B to occur as close to the occurrence of event A as possible. This problem has been studied in relation to the maximum speed of quantum computers [39] and it as found that one cannot force the system to evolve at a rate greater than the average energy.

6.5 In Which

to do this to arbitrary accuracy, ithout affecting the system. In some sense, one may not be able to determine hich ay the light cone points.

One could use the arrival of arbitrarily energetic particles in order to denote space-time events, and although one can increase the energy of the particles in order to increase the accuracy ith hich one is able to measure the order of events, at some point the energy of the particles ill effect the curvature of the neighboring space time.