3.1

ble be measured to any accuracy. On the other hand, a direct measurement

same classical time-

3.2 Free Clocks

We ill no attempt to make a measurement of the time-of-arrival. In order to do so, e ill need a clock. An ideal clock is linear in time. I.e., the position of the clock's pointer should be proportional to the time t. It is not hard to see that an ideal clock can be represented by the Hamiltonian

$$\mathbf{H}_{clock} = \mathbf{P}_{\mathbf{y}} \ . \tag{3.40}$$

To read the time of the clock, e measure the coordinate $\mathbf{y}$ conjugate to $\mathbf{P}_{\mathbf{y}}$. Using the Heisenberg equations of motion e see that the variable $\mathbf{y}$ reads the correct parameter time t found in the Schrödinger equation.

$$\begin{aligned} \mathbf{y}(t) - \mathbf{y}(t_0) &= -i \int [\mathbf{y}, \mathbf{H}_{clock}] dt \\ &= t - t_0 \end{aligned} \tag{3.41}$$

The Hamiltonian for this clock is unbounded from above and belo , nonetheless, using a sufficiently massive particle, e can approximate the ideal situation to arbitrary accuracy [1]. We rite $\mathbf{p} = \langle \mathbf{p} \rangle + \delta$

From (3.41) e see that in order to use this clock to read the time, e need to kno the initial position of the clock's dial $\mathbf{y}(t_0)$ and then subtract this from our final reading of $\mathbf{y}$. Quantum mechanics puts no limitation on ho accurately this clock can be measured. If e ant to accurately infer the time from the final reading of the clock then the clock must initially be prepared in a state ith a very small uncertainty in y. At some later point, e can measures the coordinate $y(t_f)$ to any degree of accuracy e ish to infer the time from $y(t_f) - y(t_0)$. If initially dy as very small, then e kno that the time is given by the final reading of y. Ho ever, if initially the state of the clock had a large spread in y, then the time e finally obtain ill be inaccurate by an amount dy. This means that for this clock, the inaccuracy in the time measurement is given by

$$\delta T = dy \tag{3.43}$$

If e simply ant to use this clock to read the time, then there are no restrictions on ho accurate the clock can be. So far, nothing prevents us from making the initial state of the clock's pointer as close to an eigenstate of $y(t_0)$ as e desire. Ho ever, since $\mathbf{y}(t_0)$ and $\mathbf{H}_{clock}$ do not commute (and cannot commute if the clock is to operate properly), the smaller the uncertainty in $y(t_0)$, the greater the uncertainty in H_{clock}. We ill see that if e ant to use this clock to measure the time of an event, then e ill encounter the limitation given by (3.39). We ill need to ensure that initially the position of the clock is uncertain in order for our measurements of the time of an event to succeed.

The reason for this is that since $\mathbf{y}$ is conjugate to $\mathbf{H}_{clock} = \mathbf{P_y}$, accurate clocks (hich are narro in y) have a large spread in P_y. This means that in general the energy of an accurate clock can take on fairly large values. For an infinitely accurate clock the energy ill almost al ays be infinite. Accurate clocks therefore, have a large energy uncertainty, and this makes them very hard to use to measure the time of an event. This is because accurate clocks are usually so energetic that they need a large amount of energy to turn them off. To measure the time-of-arrival of a particle, the particle itself ill have to turn off the clock hen it arrives – the external observer cannot supply any energy since she does not kno hen to turn the clock off. If the clock is much more energetic than the particle, then it ill be impossible for the particle to turn off

3.3.1 Measurement with a clock

The simplest model hich describes a direct interaction of a particle and a clock [16], ithout additional "detector" degrees of freedom, is described by the Hamiltonian

$$H = \frac{1}{2m}\mathbf{P_x}^2 + \theta(-\mathbf{x})\mathbf{P_y}. \tag{3.44}$$

Here, the particle's motion is confined to one spatial dimension, x, and $\theta(x)$ is a step function. The clock's Hamiltonian is represented by $\mathbf{P_y}$, and the time is recorded on the

On the other hand, in quantum mechanics the uncertainty relation dictates a strong back-reaction, i.e. in the limit of $\Delta y = \Delta t_A \rightarrow 0$, p_y in (3.45) must have a large uncertainty, and the state of the particle must be strongly affected by the act of measuring.

$\frac{k^2 t}{2m} + pt$. Continuity of ϕ_{kp} requires that

$$\begin{aligned} A_T &= \frac{2k}{k+q} \\ A_R &= \frac{k-q}{k+q}, \end{aligned} \tag{3.51}$$

here $q = \sqrt{k^2 + 2mp} = \sqrt{2m(E_k + p)}$.

The solution of the Schrödinger equation is

$$\psi(x, y, t) = N \int_{-\infty}^{\infty} dk \int_0^{\infty} dp f(p) g(k) \phi_{kp}(x, y, t), \tag{3.52}$$

here N is a normalization constant and $f(p)$ and $g(k)$ are some distributions. For example, ith

$$\begin{aligned} f(p) &= e^{-\Delta_y{}^2 (p-p_0)^2} \\ g(k) &= e^{-\Delta_x{}^2 (k-k_o)^2 + ikx_0}. \end{aligned} \tag{3.53}$$

and $x_0 > 0$, the particle is initially localized on the left $(x < 0)$ and the clock (ith probability close to 1) runs. The normalization in eq. (3.52) is thus $N^2 = \frac{\Delta x \Delta y}{2\pi^3}$. By choosing $p_0 \approx 1/\Delta_y$, e can no set the the clock's energy in the range $0 < p < 2/\Delta_y$.

Let us first sho that in the stationary point approximation the clock's final ave function is indeed centered around the classical time-of-arrival. Thus e assume that Δ_y and Δ_x are large such that $f(p)$ and $g(k)$ are sufficiently peaked. For $x > 0$, the integrand in (3.52) has an imaginary phase

$$\theta = qx + kx_o + py - \frac{k^2 t}{2m} - pt. \tag{3.54}$$

$\frac{d\theta}{dk} = 0$ implies

$$x_{peak}(p) = -\frac{q(k_0)}{k_0} x_o + \frac{q(k_0) t}{m}, \tag{3.55}$$

and $\frac{d\theta}{dp} = 0$ gives

$$y_{peak}(k) = t - \frac{mx}{q_0}. \tag{3.56}$$

Hence at $x = x_{peak}$ the clock coordinate y is peaked at the classical time-of-arrival

$$y = \frac{mx_o}{k_0}. \tag{3.57}$$

To see that the clock yields a reasonable record of the time-of-arrival, let us consider further the probability distribution of the clock

$$\rho(y, y)_{x>0} = \int dx |\psi(x > 0, y, t)|^2. \tag{3.58}$$

In the case of inaccurate measurements ith a small back-reaction on the particle $A_{I'} \simeq 1$. The clocks density matrix is then found (see Appendix B) to be given by:

$$\rho(y, y)_{>0} \simeq \frac{1}{\sqrt{2\pi\gamma(y)}} e^{-\frac{(y-t_c)^2}{2\gamma(y)}} \tag{3.59}$$

here the idth is $\gamma(y) = \Delta y^2 + (\frac{m\Delta x}{k_o})^2 + (\frac{y}{2k_o\Delta x})^2$. As expected, the distribution is centered around the classical time-of-arrival $t_c = x_o m/k_o$. The spread in y has a term due to the initial idth Δy in clock position y. The second and third term in $\gamma(y)$ is due to the kinematic spread in the time-of-arrival $\frac{1}{dE} = \frac{m}{kdk}$ and is given by $\frac{dx(y)\, m}{k_o}$ here $dx(y)^2 = \Delta x^2 + (\frac{y}{2m\Delta x})^2$. The y dependence in the idth in x arises because the ave function is spreading as time increases, so that at later y, the ave packet is ider. As a result, the distribution differs slightly from a Gaussian although this effect is suppressed for particles ith larger mass.

When the back-reaction causes a small disturbance to the particle, the clock records the time-of-arrival. What happens hen e ish to make more accurate measurements? Consider the exact transition probability $T = \frac{q}{k}|A_{I'}|^2$, hich also determines the probability to stop the clock. The latter is given by

$$T = \sqrt{\frac{E_k + p}{E_k}} \left[\frac{2\sqrt{E_k}}{\sqrt{E_k} + \sqrt{E_k + p}} \right]^2. \tag{3.60}$$

Since the possible values obtained by p are of the order $1/\Delta_y \equiv 1/\Delta t_A$, the probability to trigger the clock remains of order one only if

$$\bar{E}_k \delta t_A > 1. \tag{3.61}$$

Here δt_A stands for the initial uncertainty in position of the dial $\mathbf{y}$ of the clock, and is interpreted as the accuracy of the clock. $\bar{E}_k$ can be taken as the typical initial kinetic energy of the particle.

In measurements ith accuracy better then $1/\bar{E}_k$ the probability to succeed drops to zero like $\sqrt{E_k \delta t_A}$, and the time-of-arrival of most of the particles cannot be detected. Furthermore, the probability distribution of the fraction hich has been detected depends on the accuracy δt_A and can become distorted ith increased accuracy. This observation becomes apparent in the follo ing simple example. Consider an initial ave packet that is composed of a superposition of t o Gaussians centered around $k = k_1$ and $k = k_2 >> k_1$. Let the classical time-of-arrival of the t o Gaussians be t_1 and t_2 respectively. When the inequality (3.61) is satisfied, t o peaks around t_1 and t_2 ill sho up in the final probability distribution. On the other hand, for $\frac{2m}{k_1^2} > \delta t_A > \frac{2m}{k_2^2}$, the time-of-arrival of the less energetic peak ill contribute less to the distribution in y, because it is less likely to trigger the clock. Thus, the peak at t_1 ill be suppressed. Clearly, hen the precision is finer than $1/\bar{E}_k$ e shall obtain a distribution hich is considerably different from that obtained for the case $\delta t_A > 1/\bar{E}_k$ hen the t o peaks contribute equally.

3.3.2 T – $P\infty$ $_f<$

trigger ithout including the clock:

$$H_{trigger} = \frac{1}{2m}\mathbf{P}_{\mathbf{x}}^{\mathbf{2}} + \frac{\alpha}{2}(1 + \sigma_x)\delta(\mathbf{x}). \tag{3.62}$$

The particle interacts ith the repulsive Dirac delta function potential at $x = 0$, only if the spin is in the $|\uparrow_x\rangle$ state, or ith a vanishing potential if the state is $|\downarrow_x\rangle$. In the limit $\alpha \to \infty$ the potential becomes totally reflective (Alternatively, one could have considered a barrier of height α^2 and idth $1/\alpha$.) In this limit, consider a state of an incoming particle and the trigger in the "on" state: $|\psi\rangle|\uparrow_z\rangle$. This state evolves to

$$|\psi\rangle|\uparrow_z\rangle \quad \to \quad \frac{1}{\sqrt{2}}\Big[|\psi_R\rangle|\uparrow_x\rangle + |\psi_{T}\rangle|\downarrow_x\rangle\Big], \tag{3.63}$$

here ψ_R and ψ_{T} are the reflected and transmitted ave functions of the particle, respectively.

The latter equation can be re ritten as

$$\frac{1}{2}|\uparrow_z\rangle(|\psi_R\rangle + |\psi_{T}\rangle) + \frac{1}{2}|\downarrow_z\rangle(|\psi_R\rangle - |\psi_{T}\rangle) \tag{3.64}$$

Since $\uparrow_z$ denotes the "on" state of the trigger, and $\downarrow_z$ denotes the "off" state, e have flipped the trigger from the "on" state to the "off" state ith probability 1/2 [2]. Although this model only orks half the time, the chance of success does not depend in any ay on the system, and in particular, on the particle's energy. Furthermore, one can construct models here a detector is triggered almost all the time [35], although ith some energy dependence in the probability of triggering.

So far e have succeeded in recording the event of arrival to a point. We have no information at all on the time-of-arrival. It is also orth noting that the net energy exchange bet een the trigger and the particle is zero, i.e.. the particle's energy is unchanged.

$\uparrow_z$ denotes [illegible]

The eigenstates of the Hamiltonian in the basis of σ_z are

$$\Psi_L(x) = \begin{pmatrix} e^{ik_\uparrow x} + \phi_{L\uparrow} e^{-ik_\uparrow x} \\ \phi_{L\downarrow} e^{-ik_\downarrow x} \end{pmatrix} e^{ipy}, \tag{3.66}$$

for $x < 0$ and

$$\Psi_R(x) = \begin{pmatrix} \phi_{R\uparrow} e^{ik_\uparrow x} \\ \phi_{R\downarrow} e^{ik_\downarrow x} \end{pmatrix} e^{ipy}, \tag{3.67}$$

for $x > 0$. Here $k_\uparrow = \sqrt{2m(E-p)} = \sqrt{2mE_k}$ and $k_\downarrow = \sqrt{2mE} = \sqrt{2m(E_k+p)}$.

Matching conditions at $x = 0$ yields

$$\phi_{R\uparrow} = \frac{\frac{2k_\uparrow}{m} - \frac{k_\uparrow}{k_\downarrow}}{\frac{2k_\uparrow}{m} - (1 + \frac{k_\uparrow}{k_\downarrow})} \tag{3.68}$$

$$\phi_{R\downarrow} = \frac{k_\uparrow}{k_\downarrow}((\phi_{R\uparrow} - 1) = \frac{\frac{k_\uparrow}{k_\downarrow}}{\frac{2k_\uparrow}{m} - (1 + \frac{k_\uparrow}{k_\downarrow})}, \tag{3.69}$$

and

$$\phi_{L\downarrow} = \phi_{R\downarrow} \tag{3.70}$$

$$\phi_{L\uparrow} = \phi_{R\uparrow} - 1. \tag{3.71}$$

We find that in the limit $\alpha \to \infty$ the transmitted amplitude is

$$\phi_{R\downarrow} = -\phi_{R\uparrow} = \frac{\sqrt{E_k}}{\sqrt{E_k} + \sqrt{E_k + p}}. \tag{3.72}$$

P

3.3.3

There is ho ever one limiting case in hich the method does seem to succeed. Consider a narro ave peaked around k ith a idth dk. To first order in dk, the probability $T_\downarrow$ that the particle is successfully boosted is given by

$$T_\downarrow \simeq 1 - \frac{2dk}{k}. \tag{3.80}$$

Therefore in the special case that $\frac{dk}{k} << 1$, the transition probability is still close to one. If in this case e kno n in advance the value of k up to $dk << k$, e can indeed use the booster to improve the bound (3.61) on the accuracy.

The reason hy this seems to ork in this limiting case is as follo s. The probability of flipping the particle's spin depends on ho long it spends in the magnetic field described by the α term in (3.73). If ho ever, e kno beforehand, ho long the particle ill be in this field, then e can tune the strength of the magnetic field (α) so that the spin gets flipped. The requirement that $dk/k << 1$ is thus equivalent to having a small uncertainty in the "interaction time" ith this field. In some sense, the measurement is possible, because e kno the particle's momentum before hand. Of course, if e have prior kno ledge of the particle's momentum, then e could just measure $\mathbf{x}$ and argue that this allo s us to calculate the time of arrival through $t_A = mx/p$. We therefore believe that the reason the measurement procedure described above orks in this limiting case is because it assumes prior kno ledge of the particle's momentum, and e do not believe that one could improve it so that it orks for all states. These "booster" measurements cannot be used for general ave functions, and even in the special case above, one still requires some prior information of the incoming ave function.

3.3.4 Gradual triggering of the clock

In order to avoid the reflection found in the previous t o models, e shall no replace the sharp step-function interaction bet een the clock and particle by a more gradual transition.

When the WKB condition is satisfied

$$\frac{d\lambda(x)}{dx} = \epsilon << 1 \tag{3.81}$$

here $\lambda(x)^{-2} = 2m[E_0 - V(x)]$, the reflection amplitude vanishes as

$$\sim \exp(-1/\epsilon^2) \tag{3.82}$$

Solving the equation for the potential ith a given ϵ e obtain

$$V_\epsilon(x) = E_0 - \frac{1}{2m\epsilon^2}\frac{1}{x^2} \tag{3.83}$$

No e observe that any particle ith $E \geq E_0$ also satisfies the WKB condition (3.81) above for the

The problem is ho ever that the final value of $t - \mathbf{y}$ does not al ays correspond to the time-of-arrival since it contains errors due to the affect of the potential $V(x)$ on the

The time-of-arrival can hence be measured provided that $E_k \delta t >> 1$. On the other hand, hen the detector's accuracy is $\delta t < 1/E$, the particle still triggers the clock. Ho ever the measured quantity, A, no longer correspond to the time-of-arrival. Again, e see that hen e ask for too much accuracy, the particle is strongly disturbed and reading of the clock has nothing to do ith the time-of-arrival of a free particle.

3.3.5 General considerations

We have examined several models for a measurement of time-of-arrival and found a limitation,

$$\delta t_A > 1/\bar{E}_k, \tag{3.91}$$

on the accuracy that t_A can be measured. Is this limitation a general feature of quantum mechanics?

First e should notice that eq. (3.91) does not seem to follo from the uncertainty principle. Unlike the uncertainty principle, hose origin is kinematic, (3.91) follo s from the nature of the *dynamic* evolution of the system during a measurement. Furthermore here e are considering a restriction on the accuracy (not uncertainty) of a single measurement. While it is difficult to provide a general proof, in the follo ing e shall indicate hy (3.91) is expected to hold under more general circumstances.

Let us examine the basic features that gave rise to (3.91). In the toy models considered in Sections 3.3.1 and 3.3.2, the clock and the particle had to exchange energy $p_y \sim 1/\delta t_A$. As a result, the effective interaction by hich the clock s itches off, looks from the point of vie of the particle like a step function potential. This led to "non-detection" hen (3.91) as violated.

Can e avoid this energy exchange bet een the particle and the clock? Let us try to deliver this energy to some other system ithout modifying the energy of the particle.

For example consider the follo ing Hamiltonian for a clock ith a reservoir:

$$H = \frac{\mathbf{P_x}^2}{2m} + \theta(-\mathbf{x})H_c + H_{res} + V_{res}\theta(\mathbf{x}) \tag{3.92}$$

The idea is that hen the clock stops, it dumps its energy into the reservoir, hich may include many other degrees of freedom, instead of delivering it to the particle. In this model, the particle is coupled directly to the clock and reservoir, ho ever e could as ell use the idea of Section 3.3.2 above. In this case:

$$H = \frac{\mathbf{P_x}^2}{2m} + \frac{\alpha}{2}(1+\sigma_x)\delta(\mathbf{x}) + \frac{1}{2}(1+\sigma_z)H_c + H_{res} + \frac{1}{2}(1-\sigma_z)V_{res}. \tag{3.93}$$

The particle detector has the role of providing a coupling bet een the clock and reservoir.

No e notice that in order to transfer the clock's energy to the reservoir ithout affecting the free particle, e must also prepare the clock and reservoir in an initial state that satisfies the condition

$$H_c - V_{res} = 0 \tag{3.94}$$

Ho ever this condition does not commute ith the clock time variable $\mathbf{y}$. We can measure initially $\mathbf{y} - \mathbf{R}$, here R is a collective degree of freedom of the reservoir such that $[\mathbf{R}, V_{res}] = i$, but in this case e shall not gain information on the time-of-arrival y since R is unkno n. We therefore see that in the case of a sharp transition, i.e. for a localized interaction ith the particle, one cannot avoid a shift in the particle's energy. The "non-triggering" (or reflection) effect cannot be avoided.

We have also seen that the idea of boosting the particle "just before" it reaches the detector, fails in the general case. What happens in this case is that hile the detection rate increase, one generally destroys the initial information stored in the incoming ave packet. Thus although higher accuracy measurements are no possible, they do not reflect directly the time-of-arrival of the initial ave packet.